Theoretische Physik IV: Statistische Mechanik und Thermodynamik

Problem Set No. 2

Due on: Friday, 2.5.08 in the practice groups

Exercise 2.1 *(Discrete Probability Distributions)* **(10 points)**

(a) The Poisson distribution is defined for $k = 0, 1, \ldots$ by

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad \lambda > 0$$

Show that P_λ is normalized and find the expectation value of k for the Poisson distribution. (5 points)

(b) Let X_1 and X_2 be two random variables and

$$P(X_1 = x_1) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \quad x_1 = 0, 1, 2, \ldots$$

$$P(X_2 = k|X_1 = x_1) = \binom{x_1}{k} p^k (1-p)^{x_1-k} \quad k = 0, 1, \ldots, x_1$$

Show that $P(X_2 = k) = \frac{e^{-\lambda p}(\lambda p)^k}{k!}$ (5 points)

Exercise 2.2 *(Maxwell Distribution)* **(10 points)**

The velocity distribution of ideal gas atoms is given by the following probability density function:

$$f(\vec{v}) = \text{const} \cdot \exp\left(-\frac{m\vec{v}^2}{2k_BT}\right)$$

(a) Calculate the normalization constant. (2 points)

(b) Find the expectation values $\langle v_x^2 \rangle$ and $\langle \vec{v}^2 \rangle$. (4 points)

(c) What is the average value of the energy of one atom? (1 point)

(d) Calculate the probability $\rho(E)dE$ for finding an atom in the energy interval $[E, E + dE]$. (3 points)

Exercise 2.3 *(Optimized Gaussians)* **(10 points)**

Let $r_1, \ldots, r_N$ be N independent and identically distributed random variables. The probability density function for each r_i is given by $p_1(r)$. The corresponding probability distribution function be $F_1(r) = \int_{-\infty}^{r} p_1(\tilde{r}) d\tilde{r}$. Let us consider the following random variable

$$X = \max\{r_1, r_2, \ldots, r_N\}$$

which equals the largest of the N random variables r_i.

(a) Find the distribution function $F_N(x)$ of the random variable X in terms of $F_1(r)$. (2 points)

(b) Show that the most probable value x^* of the random variable X is given for $N \gg 1$ by the equation

$$p_1'(x^*) + N p_1(x^*)^2 = 0$$

(4 points)

(c) In physics there are many distributions whose tail vanishes more rapidly than a power law. Consider a probability distribution function of the r_i, where at large r we have the following relation: $\bar{F}_1(r) = 1 - F_1(r) \simeq c \exp\left(-(r/a)^p\right)$. Calculate x^* for $N \gg 1$. (4 points)

Bonus Exercise 2.4 *(Gamma Distribution)* **(10 extra points)**

Note: This exercise is optional. You may want to solve it to improve your score.

The gamma distribution is given as

$$f(x, k, b) = \frac{1}{b^k \Gamma(k)} x^{k-1} e^{-x/b} \quad x > 0, \quad k > 0, \quad b > 0$$

where $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$ is the gamma function.

(a) k is a parameter determining the shape of the distribution. Sketch the density function $f(x, k, b)$ for $b = 1$ and $k = 0.5, 1$ and 2 (using gnuplot, maple or a pen). How does the distribution change with k? (2 points)

(b) b is called a scale parameter. What happens to the distribution when k is held constant and b changes? Sketch the function f for a $k > 1$ for three different values of b. (2 points)

(c) Find the moment generating function, which is defined by $M(t) = \left\langle e^{tx} \right\rangle$ (3 points)

(d) Use the moment generating function $M(t)$ to find the mean $\langle x \rangle$ and the variance σ^2 of the distribution. (3 points)